ON THE UNRECOGNIZABILITY OF SETS

by

Ronald Barry Knode

United States Naval Postgraduate School

THESIS

ON THE UNRECOGNIZABILITY OF SETS

by

Ronald Barry Knode

June 1969

On the Unrecognizability of Sets

by

Ronald Barry Knode
Lieutenant (junior grade), United States Navy
B.S., United States Naval Academy, 1968

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE

from the

NAVAL POSTGRADUATE SCHOOL
June 1969

ABSTRACT

Let $M = \langle Q, q_0, \delta, F \rangle$ be a finite automaton over the alphabet $\Sigma = \{1, \ldots, k\}$. A state $q \in Q$ is a dead state iff $q \notin F$ and $\delta(q, \alpha) = q, \forall \alpha \in \Sigma^*$. Let $\bar{\ }$ be a mapping from Σ^* onto the non-negative integers defined by $\bar{\Lambda} = 0$ (Λ is the empty string) $\overline{\alpha\chi} = k\bar{\alpha} + \bar{\chi}$, $\chi \in \Sigma$, $\alpha \in \Sigma^*$. Define $\pi_A(n) = \#\{\alpha \in A : 0 \le \bar{\alpha} \le n\}$ and $\lambda_A(n) = \#\{\alpha \in A : \ell(\alpha) = n\}$. If A is regular let M_A be the minimal automaton recognizing A. Each automaton M induces a Markov process obtained by considering the inputs to be generated by independent rolls of a k-sided fair die. Let p(M) represent the probability of being in a final state. Let $p(A) = p(M_A)$. The following are proved: 1) $\frac{\pi_A(n)}{n} \to \theta \Rightarrow \frac{\lambda_A(n)}{2^n} \to \theta$; 2) $\frac{\pi_A(n)}{n} \to \theta$, A regular $\Rightarrow$ $p(A) = \theta$; 3) $p(A) = 0 \Rightarrow M_A$ has the dead state as the only absorbing state; 4) $\forall \varepsilon > 0\ \exists$ a regular set $A \ni M_A$ has a dead state and $\frac{\lambda_A(n+1)}{\lambda_A(n)} \to \theta$ where $k - \varepsilon < \theta < k$; 5) If $p(A) = 0$, then $\frac{\lambda_A(n+1)}{\lambda_A(n)}$ cannot converge to k. With $k = 2$, these results prove that there is no regular set A such that $\lambda_A(2n+1) = \frac{1}{n+1}\binom{2n}{n}$ and $\lambda_A(2n) = 0$. Hence there is no 1-1 mapping from the set of all trees representing expressions involving a binary + and a variable x into $\{+, x\}^*$ which preserves the number of +'s and x's and such that the set of tree images is a regular set.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

My most heartfelt thanks is extended to all those members of the Mathematics Department at the Naval Postgraduate School who instructed and inspired me to the point where this effort was both feasible and rewarding. However, my greatest "thank you" is reserved for Professor W. S. Brainerd without whose knowledge, patience, and encouragement none of this work would have been possible.

I. INTRODUCTION

Although the theory of finite automata (or sequential machines) is itself only about fifteen years old, much effort has already been devoted to exactly what automata can and cannot do. And deservedly so, for several reasons: First, and perhaps the most "pure" of the reasons, is that an automaton, by virtue of its definition, has interesting mathematical properties. Beyond this theoretical consideration, however, lie some more practical reasons. Generally speaking, an automaton is the simplest model of a digital computer. But since the number of "states" in a computer may be of the order of $2^{1000000}$, direct application of results of automata theory to an entire computer is certainly not practicable. Application of automata theory concepts and techniques is restricted to systems with relatively small numbers of states (at most in the thousands). There are, however, practical situations in which this limitation in size is met. The sequential circuits which are the basic components of computers are specified by the input-output transformation which they must realize. The circuit's operation is described in terms of states and since it must be humanly manageable, the number of states cannot be too large. In fact, any computing device, organized as an iterative array, can be separated into smaller components which will be circuits with a small number of states. Automata theory also has applications in flow-charting and program equivalence.

Many of the results achieved concerning regularity or non-regularity of sets have been done in the context of equivalence relations of finite index and a fundamental lemma concerning the way certain input sequences can be separated. Only three years ago Marvin Minsky and Seymour Papert

[10] developed a set of criteria for non-regularity based on a limiting quantity which seems intuitively to represent the portion or percentage of strings that are in the regular set. In this paper we define for regular sets, in terms of natural Markov processes, another quantity, p(A), which, generally speaking, again represents a kind of percentage of strings in the regular set and show that, when Minsky's limits exist, these two quantities agree. From the quantity developed in this paper, however, more information can be gathered concerning the machine which recognizes the regular set (and hence information about the regular set itself) than can be gathered from Minsky's criteria. Also described is a special machine which gives an upper bound for the growth rate of sets recognizable by a certain type of automaton. Finally it is shown that the set of well-formed trees written as strings is not a regular set.

II. GENERAL DISCUSSION AND DEFINITION

We will give here only as much basic automata theory as is necessary for the completion and understanding of this paper. More detail and further explanation can be found in any number of texts, such as Harrison [6], Rabin and Scott [11], or, again, Rabin [2].

Before proceeding into the definition of an automaton itself, we shall concern ourselves first with the type of input an automaton receives.

Definition 2.1 An alphabet Σ is a finite set of symbols.

Definition 2.2 A tape α is any finite sequence of symbols from the alphabet. Included also as a tape is the empty tape, denoted Λ, which is the tape with no symbols. Defined on the set of tapes is the operation of concatenation, or juxtaposition, i.e., if α is a tape and β is a tape, then $\alpha . \beta$ is the tape formed by "concatenating" α and β. If R is a set of tapes and S is a set of tapes then R.S = $\{\alpha . \beta : \alpha \in R \text{ and } \beta \in S\}$.

Definition 2.3 $R^0 = \{\Lambda\}$, $R^{n+1} = R^n . R$

Definition 2.4 $R^* = \bigcup_{i=0}^{\infty} R^i$

Thus we see that the set of all tapes from the alphabet Σ is Σ^*. With this operation of concatenation defined as above, $\langle \Sigma^*, \Lambda, . \rangle$ is a monoid, and in fact, Σ^* is the free monoid generated by Σ.

For this paper we shall let $\Sigma = \{1, 2\}$, although the results are true for any finite alphabet $\{1, 2, 3, \ldots, k\}$. The prime motivation for this particular Σ is that it is the simplest case having all the properties of the general case. The additional fact that most real-life machines are binary oriented is another consideration in choosing the two element alphabet. Then Σ^* is the collection of all possible sequences of 1's and 2's, including the empty string Λ . Here the reader must be careful not to associate any numerical significance to the symbol "1" or the symbol "2". These symbols are merely inputs to the machine which we now shall define.

Definition 2.5 A (finite) automaton over the alphabet Σ is a quadruplet $M = \langle Q, q_0, \delta, F \rangle$ where Q is a finite non-empty set (the set of "states"), q_0 is an element of Q (the initial state), δ is a mapping of $Q \times \Sigma$ into Q (the state transition function), and F is a subset of Q (the set of "final" or "accept" states).

The state transition function δ can be extended from $Q \times \Sigma$ to $Q \times \Sigma^*$ in a very natural way by recursive definition as follows:

$$\delta(q, \Lambda) = q \qquad \text{for all } q \in Q$$

$$\delta(q, \alpha . x) = \delta(\delta(q, \alpha), x) \quad \text{for } \alpha \in \Sigma^*,\ x \in \Sigma,\ q \in Q$$

Definition 2.6 The set of tapes accepted or defined by the automaton M, denoted T(M), is the collection of all tapes α in Σ^* such that $\delta(q_0, \alpha)$ is an element of F.

Definition 2.7 A set R is called regular (or recognizable) if and only if there exists a finite automaton M such that R = T(M).

The preceding definition merely says that a set R is regular only if there is an automaton M which accepts all tapes in R and rejects all others, i.e., $\delta(q_0, \alpha) \in F$ iff $\alpha \in R$.

Recall now that Σ has been defined as the set $\{1, 2\}$. In order to enable us to lend some numerical bearing to this research we define the following one-to-one mapping from $\{1, 2\}^*$ onto the non-negative integers.

$$\bar{\ } : \{1,2\}^* \longrightarrow \text{Non-negative integers}$$

By establishing a sort of alphabetical hierarchy between 1 and 2 in Σ we can define the "bar" mapping as:

$$\overline{\Lambda} = 0$$
$$\overline{1} = 1$$
$$\overline{2} = 2$$
$$\overline{11} = 3$$
$$\overline{12} = 4$$
$$\overline{21} = 5$$
$$\overline{22} = 6$$
$$\overline{111} = 7$$
$$\vdots$$

Thus we now have associated with each member of Σ^* a non-negative integer, and vice-versa. We call the "bar" mapping the <u>encoding</u> mapping and its inverse the <u>decoding</u> mapping.

As the last general definition, we define a natural relation in the monoid $\langle \Sigma^*, \Lambda, . \rangle$.

Definition 2.8 Let $\alpha, \beta \in \Sigma^*$. Then $\alpha < \beta \Leftrightarrow \exists \gamma \in \Sigma^*$ $\ni \beta = \alpha . \gamma$

III. LIMIT THEOREMS

Let A be a subset of $\{1, 2\}^*$. The question here is what kind of automaton could "recognize" A in the sense of definition 2.7? Since for any regular set A there can be any number of automata M such that A = T(M), we shall concern ourselves only with the unique minimal automaton recognizing A and we will call this automaton M_A.

Definition 3.1 $\pi_A(n)$ = the cardinality of the set $\{\alpha \in A : 0 \leq \bar{\alpha} \leq n\}$

Definition 3.2 Let $\alpha \in \{1,2\}^*$. The length of α, $\ell(\alpha)$ is the number of digits in α. Then we have $\sum_{i=0}^{\ell(\alpha)-1} 2^i \leq \bar{\alpha} < \sum_{i=0}^{\ell(\alpha)}$ Furthermore,

$$\overline{\alpha.\beta} = 2^{\ell(\beta)} \cdot \bar{\alpha} + \bar{\beta}, \quad \alpha, \beta \in \Sigma^*.$$

Definition 3.3 $\lambda_A(n)$ = the cardinality of the set $\{\alpha \in A : \ell(\alpha) = n\}$

Definition 3.4 A dead state of M_A is a state $q \in Q$ such that $\delta(q, \alpha) \in F$ is satisfied by no $\alpha \in \{1,2\}^*$. Since there is only one dead state in the minimal automaton, this is equivalent to saying $\delta(q, \alpha) = q$ for all $\alpha \in \{1,2\}^*$.

The next three theorems are generalizations and small extensions of the work done by Minsky and Papert in [10], and are stated without proof since the proofs are, with only minor modifications for theorems 3.5 and 3.8, essentially identical with those in [10]. The first two theorems are concerned with the consequences of M_A having a dead state, and the last theorem indicates what must happen if M_A has no dead state.

Theorem 3.5 Let $M = M_A$ and suppose:

(a) $\delta(q_0, \alpha)$ is dead

(b) $\xi_0 = \dfrac{\bar{\alpha} + 1/2}{\bar{\alpha}}$

(c) $\lim\limits_{n \to \infty} \dfrac{\pi_A(n)}{\pi_A(\xi_0 n)} = \theta$

Conclusion: $\theta = 1$

From theorem 3.5 we get immediately

Corollary 3.6 If $\left\{\frac{\pi_A(n)}{\pi_A(\xi n)}\right\} \longrightarrow \theta(\xi)$ for all real ξ and if $\theta(\xi)=1 \Leftrightarrow \xi=1$, then A cannot be a regular set whose minimal automaton has a dead state.

In some cases the sequence $\left\{\frac{\pi_A(n)}{\pi_A(\xi n)}\right\}$ fails to converge, in which case we can still sometimes use the following theorem.

Theorem 3.7 Let α_r be the rth member of A in order of magnitude under the encoding mapping. That is, there are r-1 members of A whose image under the encoding mapping is less than $\overline{\alpha_r}$. Then if

$$\lim_{r\to\infty} \frac{\overline{\alpha_{r+1}} - \overline{\alpha_r}}{\overline{\alpha_r}} = 0$$

A cannot be a regular set whose minimal automaton has a dead state.

Theorem 3.8 If A is regular and M_A has no dead state, then $\frac{\pi_A(n)}{n} \geqslant 2^{-4N}$ where N is the number of states of M_A. Thus the density of A cannot converge to zero.

Combining these results yields the following criterion.

Criterion: To prove that a set, A, is not regular, it is sufficient to verify condition 1 and condition 2 or 3.

Condition 1. $\frac{\pi_A(n)}{n} \to 0$ as $n \to \infty$

Condition 2. $\frac{\pi_A(n)}{\pi_A(\xi n)} \to \theta(\xi)$ as $n\to\infty$, and $\theta(\xi)\neq 1 \ \forall \xi \neq 1$

Condition 3. $\frac{\overline{\alpha_{n+1}} - \overline{\alpha_n}}{\overline{\alpha_n}} \to 0$ as $n \to \infty$

If A is regular, then by theorem 3.8 M_A has a dead state, but by corollary 3.6 or theorem 3.7 it has none. Thus we are led quickly to a contradiction.

IV. THE PROBABILITY OF ACCEPTANCE

When using limit theorems as the ones developed in section three we run into several problems. First, of course, is the nature of the limiting process itself. The limit may or may not exist, and, even if we can establish its existence, we may not be able to determine its value. Secondly, unless $\pi_A(n)$ is a relatively easy sequence to recognize, we cannot begin to evaluate the needed sequences in the theorems. If $\lim_{n\to\infty} \frac{\pi_A(n)}{n}$ exists, it can be loosely interpreted as the percentage of strings of $\{1,2\}^*$ which are in the set in question, namely A, and hence as some sort of probability that a string is in A. If A is regular, this reduces to a probability that a string is accepted. The above interpretation is all on an intuitive basis and will not be subjected to any rigorous analysis. In this section an algorithm is given which defines another quantity which, again loosely, can be interpreted as the probability that a string is accepted.

If we assume that the set A is regular, then M_A has a finite number of states. Consider the transition function of M_A, δ, written as a transition matrix, i.e., if the machine is in state q_i, then with probability 1/2 we apply to q_i an input of a "1", and with probability 1/2 we apply an input of a "2". This is not to say that the machine is no longer working deterministically, but merely that the input is generated by a sequence of Bernoulli trials with the probability of a "1" = the probability of a "2" = 1/2. If we were dealing with a k-symbol alphabet $\{1,2,\ldots,k\}$, then the probabilities would have been 1/k. By evaluating $\delta(q_i,1)$ and $\delta(q_i,2)$ we can establish which states are accessible from q_i with an input of length one. Assume M_A has N states.

Then by evaluating $\delta(q_i,1)$ and $\delta(q_i,2)$ for $i = 0,1,2,\ldots,N-1$ and arranging the states of M_A in a square pattern viz.,

$$\begin{array}{ccccc} & q_{N-1} & q_{N-2} & \cdots & q_0 \\ q_{N-1} & & & & \\ q_{N-2} & & & & \\ \vdots & & & & \\ q_0 & & & & \end{array}$$

and letting the q_{ij}th entry in this pattern be the "probability" that $\delta(q_i,\alpha) = q_j$ given that the length of α is one, we get a matrix of values that begins to look suspiciously like a Markov chain. Upon further inspection, we see that all entries are either 0 ($\delta(q_i,\alpha) = q_j$ for neither $\alpha = 1$ nor $\alpha = 2$), 1/2 ($\delta(q_i,\alpha) = q_j$ for either $\alpha = 1$ or $\alpha = 2$), or 1 ($\delta(q_i,\alpha) = q_j$ for both $\alpha = 1$ and $\alpha = 2$). Finally it is noted that $\sum_{j=0}^{N-1} q_{ij} = 1$, $i = 0,1,2,\ldots,N-1$. Thus we do indeed have a Markov chain. Let us call this matrix M, since it describes somewhat the original automaton. Clearly this matrix does not define the automaton since the transition function can only be interpreted generally from this matrix and we have no information whatsoever on which states are final states. However, we can glean some information from this approach.

From Feller [3], we know that M can be divided, in a unique manner, into closed sets, $C_1, C_2, \ldots, C_k$, such that from any state of a given set all states of that set, and no other, can be reached. We shall use, without ambiguity, the notation C_i to represent both a block matrix in M and the set of states making up that block matrix. M can now be rewritten:

$$M = \begin{pmatrix} (C_1) & & & 0 & \\ & (C_2) & & & \\ 0 & & \ddots & & \\ & & & (C_k) & 0 \; 0 \cdots \\ & & & & \\ g_{\cdot j} \to & & & & \\ g_{\cdot j} \to & & & & \end{pmatrix}$$

Since each C_j can be treated independently as a Markov chain (all entries to the right and left of each individual C_j are zero entries), and each C_j is a closed set, then each C_j is ergodic and the stationary (ergodic) probabilities can be calculated. These stationary probabilities are merely the probabilities of being in the various states of C_j after a large number of steps given that the process was already in a state of C_j. The results of the ergodic theorem [5] make these probabilities independent of the starting state in C_j, i.e., if $q_a \in C_k$, and p_a is the stationary probability for state q_a, then $p_a = \lim_{n\to\infty} (C_k{}^n)_{ia}$ for all i where $(C_k)_{ia}$ is the iath entry of the matrix C_k. In general, M will also contain transient states, i.e., states not in any C_j, from which states of the closed sets can be reached, but not vice-versa. These transient states are carriers, taking the process from its beginning to one of the closed sets.

Let us look more closely at these closed sets C_j within the context of the entire Markov chain M. Since each C_j is closed, i.e., we can never leave C_j once we are in a state contained in C_j, each C_i, i = 1, 2,...,k <u>taken as a whole</u> is an absorbing state in the original Markov chain M. This means that if the process begins in some transient state and travels eventually to a state in C_j, then the process has been "absorbed" into C_j never again to leave. This may be better shown if we let c_i be the number of states in C_i, $i = 1,2,\ldots,k$, and associate with M a matrix M' where

$$M' = \begin{pmatrix} (I_{c_1}) & & & 0 & \\ & (I_{c_2}) & & & \\ 0 & & \ddots & & \\ & & & (I_{c_k}) & 0\ 0 \cdots \\ & & & & \\ q_{ij} \rightarrow & & & & \\ q_{oj} \rightarrow & & & & \end{pmatrix}$$

The associated matrix M' is the original matrix M with each C_j replaced by a $c_j \times c_j$ identity matrix I_{c_j}. If we let $r = c_1 + c_2 + \ldots + c_k$ and let $s = N-r$, then M' has the canonical form

$$M' = \left(\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right)$$

where I is an r x r identity matrix, O is an r x s zero matrix, R is an s x r matrix, and Q is an s x s matrix. The canonical form of M' is recognized more readily as an absorbing Markov chain.

Now $(M')^n$ gives the probabilities of being in the various states starting from various states after n steps. From Kemeny and Snell [7], we know that $(M')^n$ can be written

$$(M')^n = \left(\begin{array}{c|c} I & 0 \\ \hline R_n & Q^n \end{array}\right)$$

where R_n is merely the s x r matrix of $(M')^n$ after we have blocked off I, O, Q^n in the manner shown. (Note that R_n is not necessarily R^n). This form shows that the entries in Q^n give the probabilities of being in each non-absorbing state after n steps for each possible non-absorbing starting state. By Kemeny [7], we know that the probability that the process will be absorbed is one. Therefore each entry in Q^n must approach zero as n approaches infinity, which says that $Q^n \rightarrow 0$ as $n \rightarrow \infty$. (After zero steps the process is in the same non-absorbing state in which it started. Hence $Q^0 = I$.) But $Q^n \rightarrow 0$ is a sufficient condition for I-Q to be non-singular [8]. Let $K = (I-Q)^{-1}$. We call K the <u>fundamental matrix</u> for the given absorbing chain.

Following closely a procedure given by Kemeny and Snell [7], we will now describe an algorithm which will compute a quantity which we call p(A) that is an indicator of the type of automaton which recognizes A and represents, in a sense, the probability that a string is accepted. Furthermore, if A is regular, p(A) will always exist.

Given a regular set A and the minimal automaton recognizing A, M_A, construct the state transition matrix M from the transition function. For ease of computation put the q_{0i}th row as the last row of the M matrix. Find the closed sets, the C_i, in M. Compute the associated matrix M' and put M' in the canonical form. Next compute $K = (I-Q)^{-1}$, and let B = KR, where Q and R are as in the canonical form of M'. The entries in the bottom row of B (the q_0 row) will give us the probabilities of ending up in any particular absorption state. These probabilities are the absorption probabilities. For each C_j of M, sum the absorption probabilities in the corresponding I_{c_j} of M'. These sums are the probabilities of ending up in any particular closed set C_j. Now either by computing $\lim (C_j^n)_{ia}$ for some i and j or by solving the following system of equations, find the stationary probability for each non-transient state.

$$\left.\begin{aligned} &\text{Equations 4.1 (a)} \quad p_j = \sum_{m=1}^{c_i} p_m (C_i)_{mj} \\ &j = 1,2,\ldots,c_i \text{ where } p_j \text{ is the stationary probability for state } q_j. \\ &\text{(b)} \quad \sum_{m=1}^{c_i} p_m = 1 \end{aligned}\right\} i = 1,2,\ldots,k$$

Note that in the above system, for each i, there are $c_i + 1$ equations in c_i unknowns. Equation (b) is necessary since there are only $c_i - 1$ independent equations in (a). Multiplying the stationary probabilities of C_j by the sum of the absorption probabilities for that C_j yields the final probability of being in each particular state. Now sum these final probabilities over all $q_i \in F$ and call this sum p(A). The procedure we have just described has been for the general case where q_0 is a transient state. If q_0 is not a transient state, then this implies that M itself is ergodic (otherwise we would not have the minimal automaton) and the procedure reduces to finding the stationary probabilities and summing the stationary probabilities for all $q_i \in F$, i.e., solving equations 4.1 with $k = 1$ and $c_1 = N$, the number of states in M_A, and summing the correct probabilities. Finally we condense this description into

Algorithm 4.2 Given a regular set A, and the minimal automaton recognizing A, M_A,

$$p(A) = \sum_{q_a \in F} p_a'$$

where

$$p_a' = \left(\sum_{q_i \in C_a} (KR)_{0i} \right) \cdot p_a$$

and p_a is found by solving system 4.1.

At this point one may ask what if a transient state is an element of F? But since $Q^n \to 0$, the probability of ending up in any transient state is zero, and hence this state would add nothing to p(A). As an example for algorithm 4.2, we will compute p(A) for the following automaton: $M_A = \langle Q, q_0, \delta, F \rangle$ where

$Q = \{q_0, q_1, q_2, q_3, q_4\}$, $F = \{q_1, q_2, q_4\}$, and $\delta(q_0, 1) = q_2$, $\delta(q_0, 2) = q_1$, $\delta(q_1, 1) = q_3$, $\delta(q_1, 2) = q_2$, $\delta(q_2, 1) = q_2$, $\delta(q_2, 2) = q_2$, $\delta(q_3, 1) = q_4$, $\delta(q_3, 2) = q_3$, $\delta(q_4, 1) = q_4$, $\delta(q_4, 2) = q_3$.

M for this automation is,

$$
M = \begin{array}{c c}
\begin{matrix} \quad q_4 & \quad q_3 & \quad q_2 & q_1 & q_0 \end{matrix} & \\
\begin{pmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0
\end{pmatrix} & \begin{matrix} q_4 \\ q_3 \\ q_2 \\ q_1 \\ q_0 \end{matrix}
\end{array}
$$

Furthermore, we find that

$$
C_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad C_2 = (1)
$$

$$
R = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}
$$

$$
I - Q = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix} \qquad K = (I - Q)^{-1} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}
$$

Finally we calculate,

$$K\cdot R = \begin{pmatrix}1 & 0\\ 1/2 & 1\end{pmatrix}\begin{pmatrix}0 & 1/2 & 1/2\\ 0 & 0 & 1/2\end{pmatrix} = \begin{array}{c}\begin{pmatrix}0 & 1/2 & 1/2\\ 0 & 1/4 & 3/4\end{pmatrix}\\ \begin{array}{ccc}q_4 & q_3 & q_2\end{array}\end{array}$$

Looking at the bottom row of KR we find that the probability of ending up in C_1 is $0 + 1/4 = 1/4$, and the probability of ending up in C_2 is $3/4$. Next we compute the stationary probabilities in C_1 and C_2, and find that the stationary probabilities for q_2, q_3, and q_4 are 1, 1/2, and 1/2 respectively. Therefore, the probability of ending up in q_2 is $1 \cdot 3/4 = 3/4$, in q_3 is $1/4 \cdot 1/2 = 1/8$, and in q_4 is $1/4 \cdot 1/2 = 1/8$. Summing over the states in F, namely q_2 and q_4, we find that $p(A) = 3/4 + 1/8 = 7/8$.

From the definition and construction of p(A) we obtain the following theorem.

Theorem 4.3 If A is regular and $p(A) = 0$, then M_A must have a dead state. Furthermore, the dead state is the only absorbing state in M_A and the transition matrix M is an absorbing Markov chain with one absorbing state, namely the dead state.

Proof: First of all, the assumption that $p(A) = 0$ implies that the matrix M itself cannot be ergodic. If M were ergodic, we could reach every state from every other state and from Fisz [5] we would have all stationary probabilities positive. But $p(A) = 0$ implies that the stationary probabilities for states of F are zero. Hence, we have a contradiction. Therefore, consider the transition matrix of M_A in the block form

$$M = \begin{pmatrix} (C_1) & & & & \\ & (C_2) & & 0 & \\ 0 & & \ddots & & \\ & & & (C_k) & 0 \\ q_{ij} \rightarrow & & & & \\ q_{0j} \rightarrow & & & & \end{pmatrix}$$

Let us first consider each C_k. Again from Fisz since each C_k is a closed set and every state in C_k can be reached from every other state in C_k, all stationary probabilities in each C_k are positive. Since the machine is minimal, each state is accessible which means that the probability of being absorbed by each C_k is positive. Hence, no state in any C_k is in F because otherwise p(A) would be positive. This means that we have remaining only C_k's with states not in F (and there must be at least one such C_k since the probability that the process is absorbed is one). Since these C_k's contain only states not in F and are closed, i.e., if $q \in C_k$, $\nexists\, \alpha \in \{1,2\}^* \ni \delta(q,\alpha) \in F$, each of these states is dead. Because M_A is the minimal machine, there can be at most one dead state.

Corollary 4.4 If A is regular and p(A) = 0, then for every $\alpha \in \Sigma^* \; \exists\, \beta \in \Sigma^* \ni \delta(q_0, \alpha.\beta) = q_D$ where q_D is the dead state of M_A.

Proof: From theorem 4.3 we know that the transition matrix of M_A has the form

$$M = \left(\begin{array}{c|c} 1 & 0 \\ \hline R & Q \end{array}\right)$$

where the lone absorbing state is the dead state. We know also that the probability that the process is absorbed is one. This means that from any state of M_A we must be able to get to the dead state. Therefore, for any $\alpha \in \Sigma^* \ \exists \ \beta \in \Sigma^* \ni \delta[\delta(q_0,\alpha),\beta] = \delta(q_0,\alpha.\beta) = q_D$.

V. THE AGREEMENT OF THE PROBABILITIES

Thus far we have presented two quantities, $\lim_{n\to\infty} \frac{\pi_A(n)}{n}$, and p(A), both of which seem intuitively to represent the probability that a string is accepted. On the one hand $\lim_{n\to\infty} \frac{\pi_A(n)}{n}$ exists for some regular and some non-regular sets, while p(A) exists only and always for regular sets. In order for the conclusions drawn from both these quantities to be truly valid and meaningful, they should agree where they both exist. In this section we show that this is indeed the case, i.e., if A is regular and $\lim_{n\to\infty} \frac{\pi_A(n)}{n} = \theta$, then $\theta = p(A)$. Before we can show this, however, we need the following lemma.

Lemma 5.1 If $\frac{\pi_A(n)}{n} \to \theta$, then $\frac{\lambda_A(n)}{2^n} \to \theta$.

Proof: Suppose $\frac{\lambda_A(n)}{2^n}$ doesn't converge to θ. Then $\exists\, \varepsilon > 0$ such that for infinitely many n $\frac{\lambda_A(n)}{2^n} > \theta + \varepsilon$ (or $< \theta - \varepsilon$). Let M be this set of n's. Now let $\alpha_n = 2^n$ (the string, not the number). Since $\frac{\bar{\alpha}}{\bar{\alpha}+1} \to 1 = \frac{\bar{\alpha}}{\bar{\alpha}}$ $\exists N_1 \ni$ for all n $\geq N_1$ $\left|\frac{\bar{\alpha}_n}{\bar{\alpha}_n} - \frac{\bar{\alpha}_n}{\bar{\alpha}_n+1}\right| < \varepsilon/4$. Similarly, since $\frac{2^n}{2^n-1} \to 1 = \frac{2^n}{2^n}, \exists N_2 \ni$ for all n $\geq N_2$ $\left|\frac{2^n}{2^n} - \frac{2^n}{2^n-1}\right| < \varepsilon/4$. From the hypothesis, $\exists N_3 \ni$ for all n $\geq N_3$ $\left|\frac{\pi_A(\bar{\alpha}_n)}{\bar{\alpha}_n} - \theta\right| < \varepsilon/4$. Choose n $> N + 1$ where N = max (N_1, N_2, N_3) and such that n $\in$ M. Again let $\alpha_n = 2^n$. Now

$$\frac{\pi_A(\bar{\alpha}_n)}{\bar{\alpha}_n} = \frac{\pi_A(\bar{\alpha}_{n-1}) + \lambda_A(n)}{\sum_{k=1}^{n} 2^k} = \frac{\pi_A(\bar{\alpha}_{n-1}) + \lambda_A(n)}{2^{n+1} - 2}$$

$$= \frac{\pi_A(\bar{\alpha}_{n-1})}{2^{n+1}-2} + \frac{\lambda_A(n)}{2^{n+1}-2}$$

$$= \frac{1}{2}\left[\frac{\pi_A(\bar{\alpha}_{n-1})}{(2^n-2)+1} + \frac{\lambda_A(n)}{2^n-1}\right]$$

Since $0 \leqslant \dfrac{\pi_A(\bar{\alpha}_{n-1})}{\bar{\alpha}_{n-1}} \leq 1$, $n-1 > N \Rightarrow$

$$\Rightarrow \frac{\pi_A(\bar{\alpha}_{n-1})}{\bar{\alpha}_{n-1}} \cdot \left| \frac{\bar{\alpha}_{n-1}}{\bar{\alpha}_{n-1}} - \frac{\bar{\alpha}_{n-1}}{\bar{\alpha}_{n-1}+1} \right| \leq \varepsilon/4$$

$$\Rightarrow \left| \frac{\pi_A(\bar{\alpha}_{n-1})}{\bar{\alpha}_{n-1}} - \frac{\pi_A(\bar{\alpha}_{n-1})}{\bar{\alpha}_{n-1}+1} \right| \leq \varepsilon/4$$

Similarly, since $0 \leqslant \dfrac{\lambda_A(n)}{2^n} \leq 1$, $n > N \Rightarrow$

$$\Rightarrow \frac{\lambda_A(n)}{2^n} \cdot \left| \frac{2^n}{2^n} - \frac{2^n}{2^n-1} \right| \leq \varepsilon/4$$

$$\Rightarrow \left| \frac{\lambda_A(n)}{2^n} - \frac{\lambda_A(n)}{2^n-1} \right| \leq \varepsilon/4$$

Therefore, for n $> N + 1$

$$\frac{\pi_A(\bar{\alpha}_n)}{\bar{\alpha}_n} \geq \frac{1}{2}\left[\frac{\pi_A(\bar{\alpha}_{n-1})}{\bar{\alpha}_{n-1}} - \varepsilon/4 + \frac{\lambda_A(n)}{2^n} - \varepsilon/4\right]$$

Thus for infinitely many n,

$$\frac{\pi_A(\bar{\alpha}_n)}{\bar{\alpha}_n} \geq \frac{1}{2}\left[\theta - \varepsilon/4 - \varepsilon/4 + \theta + \varepsilon - \varepsilon/4\right]$$

$$\geq \frac{1}{2}\left[2\theta + \varepsilon/4\right] = \theta + \varepsilon/8$$

But this contradicts the hypothesis that $\frac{\pi_A(n)}{n}$ converges to θ .

Theorem 5.2 If A is regular and $\frac{\lambda_A(n)}{2^n} \to \theta$, then $\theta = p(A)$.

Proof: First we define $\lambda_a(n)$ to be the cardinality of the set $\{\alpha \in \{1,2\}^* : \delta(q_0,\alpha) = q_a \text{ and } \ell(\alpha) = n\}$. Then clearly $\lambda_A(n) = \sum_{q_a \in F} \lambda_a(n)$. If we consider the transition matrix of M_A we have two cases.

Case 1. M itself is ergodic. Then the probability of capture in state q_a in n steps is $\frac{\lambda_a(n)}{2^n} = (M^n)_{0a}$. Taking limits on both sides we have, from Fisz [5], $\lim_{n\to\infty} \frac{\lambda_a(n)}{2^n} = \lim_{n\to\infty} (M^n)_{0a} = p_a$, where p_a is the stationary probability for state q_a. Finally,

$$\theta = \lim_{n\to\infty} \frac{\lambda_A(n)}{2^n} = \lim_{n\to\infty} \sum_{q_a\in F} \frac{\lambda_a(n)}{2^n}$$

$$= \sum_{q_a\in F} \lim_{n\to\infty} \frac{\lambda_a(n)}{2^n}$$

$$= \sum_{q_a\in F} p_a = p(A).$$

Case 2. M is not ergodic. In this case, as in algorithm 4.1, we rewrite M in the form

$$M = \begin{pmatrix} (C_1) & & & & \\ & (C_2) & & 0 & \\ & & \ddots & & \\ 0 & & & (C_k) & 0 \\ q_{ij} \rightarrow & & & & \\ q_{oj} \rightarrow & & & & \end{pmatrix}$$

If q_a is not in any C_i, then $\frac{\lambda_a(n)}{2^n} = (Q^n)_{oa}$ and $\lim_{n\to\infty} \frac{\lambda_a(n)}{2^n} = \lim_{n\to\infty} (Q^n)_{oa} = 0$. Hence we can eliminate consideration of all states of F which are also transient states. Assume now that $q_a \in C_a$. Let T be the set of transient states. Then the probability of capture in state q_a in n steps is

$$\frac{\lambda_a(n)}{2^n} = \sum_{k=1}^{n} \left\{ \sum_{t\in C_a} \left[\sum_{s\in T} (Q^{n-k})_{os} (R)_{st} \right] (M^{k-1})_{ta} \right\}$$

Fix $t \in C_a$, and $s \in T \ni (R)_{st} \neq 0$. If $(R)_{st} = 0$, then this path adds nothing to the sum of all paths which have length n and end in state q_a. If $(R)_{st} = 0$ for all s and t, then we do not have the minimal machine, since this would mean none of the states in C_a are accessible. Therefore, there exists at least one combination s, t such that $(R)_{st} \neq 0$. Let P_n be the probability of capture in state q_a in n steps via this certain path. Then

$$P_n = \sum_{k=1}^{n} (Q^{n-k})_{os} (R)_{st} (M^{k-1})_{ta}$$

$$= (Q^{n-1})_{os}(R)_{st}(M^{0})_{ta} + \cdots + (Q^{0})_{os}(R)_{st}(M^{n-1})_{ta}$$

We know the following things:

(i) $\lim_{n\to\infty} \sum_{k=0}^{n} (Q^{k})_{os} = (I-Q)^{-1}_{os}$ for all s from Finkbeiner [4].

Furthermore $(I-Q)^{-1}_{os} = \sum_{k=0}^{\infty}(Q^{k})_{os} \neq 0$, or else state q_s would not be accessible.

(ii) Since $(M^{n})_{ta} = (C_a{}^{n})_{ta}$

$$\lim_{n\to\infty}(M^{n})_{ta} = p_a$$

(iii) $\lim_{n\to\infty}(Q^{n})_{os} = 0$ for all s

Given $\varepsilon > 0$,

(a) Choose $N_3 \ni$ for $n \geq N_3$

$$\left| \sum_{k=0}^{n}(Q^{k})_{os} - (I-Q)^{-1}_{os} \right| < \varepsilon/(R)_{st}$$

(b) Choose $N_2 \ni$ for $n \geq N_2$

$$\left| (M^{n})_{ta} - p_a \right| < \varepsilon/(I-Q)^{-1}_{os}$$

(c) Choose $N_1 \ni$ for $n \geq N_1$

$$(Q^{n})_{os} < \frac{\varepsilon}{(R)_{st}GN_2}$$

where

$$G = \max\{(M^{0})_{ta}, (M^{1})_{ta}, \ldots, (M^{N_2-1})_{ta}\}$$

Let $n > N_1 + N_2 + N_3 + 1$.

Let $$B_n = \left[(Q^{n-1})_{os}(R)_{st}(M^{0})_{ta} + \cdots + (Q^{n-N_2})_{os}(R)_{st}(M^{N_2-1})_{ta}\right]$$

and

$$D_n = \left[\left(Q^{n-(N_2+1)}\right)_{os} (R)_{st} + \cdots + \left(Q^0\right)_{os} (R)_{st} \right]$$

By (c) since $n-N_2 > N_1$

$$0 \leq B_n \leq \sum_{k=n-N_2}^{n-1} \left(Q^k\right)_{os} (R)_{st} \cdot G$$

$$< N_2 \cdot \frac{\varepsilon}{(R)_{st} G N_2} \cdot (R)_{st} \cdot G = \varepsilon$$

Then, using (b) also

$$D_n p_a - D_n \cdot \frac{\varepsilon}{(I-Q)^{-1}_{os}} \leq P_n \leq \varepsilon + D_n p_a + D_n \frac{\varepsilon}{(I-Q)^{-1}_{os}}$$

By (a), since $n-(N_2 + 1) > N_1 + N_3 \geq N_3$

$$D_n p_a - \left[(I-Q)^{-1}_{os} + \frac{\varepsilon}{(R)_{st}} \right] \frac{\varepsilon}{(I-Q)^{-1}_{os}} \leq P_n$$

$$P_n \leq D_n p_a + \varepsilon + \left[(I-Q)^{-1}_{os} + \frac{\varepsilon}{(R)_{st}} \right] \frac{\varepsilon}{(I-Q)^{-1}_{os}}$$

or

$$D_n p_a - \varepsilon - \xi \leq P_n \leq D_n p_a + 2\varepsilon + \xi$$

Where

$$\xi = \frac{\varepsilon^2}{(I-Q)^{-1}_{os} (R)_{st}}$$

Since ε can be made arbitrarily small, we have

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} D_n \rho_a = \lim_{n\to\infty}\left[\sum_{k=0}^{n} (Q^k)_{os} (R)_{st}\right]\rho_a$$

$$= (I-Q)^{-1}_{os} (R)_{st}\rho_a = (K)_{os} (R)_{st}\rho_a$$

And finally we arrive at

$$\lim_{n\to\infty} \frac{\lambda_A(n)}{2^n} = \lim_{n\to\infty} \sum_{q_a\in F} \frac{\lambda_a(n)}{2^n}$$

$$= \lim_{n\to\infty} \sum_{q_a\in F} \sum_{t\in C_a} \sum_{s\in T} (P_n)_{ast}$$

$$= \sum_{q_a\in F} \sum_{t\in C_a} \sum_{s\in T} \lim_{n\to\infty} (P_n)_{ast}$$

$$= \sum_{q_a\in F} \sum_{t\in C_a} \sum_{s\in T} (K)_{os} (R)_{st}\rho_a$$

$$= p(A)$$

The reader is advised to compare this with algorithm 4.2 in which we described the construction of p(A).

Theorem 5.3 If A is regular and $\frac{\pi_A(n)}{n} \to \theta$, then θ = p(A).

Proof: Apply lemma 5.1 and theorem 5.2.

VI. THE FIBONACCI MACHINE--A GROWTH RATE UPPER BOUND

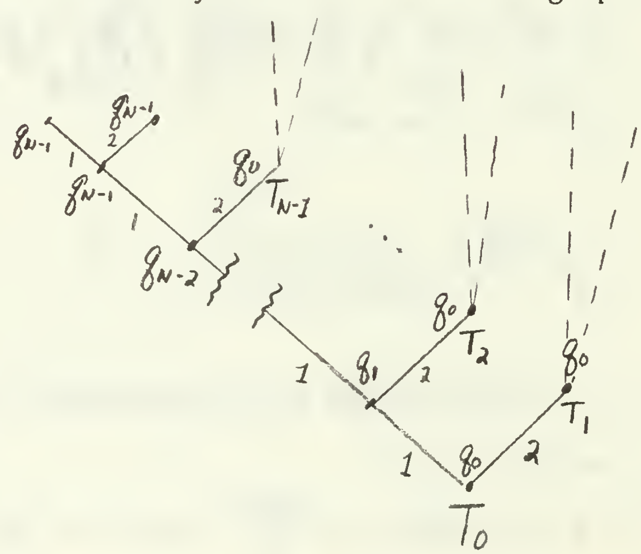
For the proof of the next theorem, it becomes necessary to look at a particular machine of N states, one of which is a dead state, which we shall call M_N. Described formally, $M_N = \langle Q, q_0, \delta, F \rangle$ where

$$Q = \{q_0, q_1, q_2, \ldots, q_{N-1}\}$$
$$F = \{q_0, q_1, \ldots, q_{N-2}\}$$

and the transition function is given by

$$\delta(q_i, 2) = q_0, \qquad i = 0, 1, \ldots, N-2$$
$$\delta(q_i, 1) = q_{i+1}, \qquad i = 0, 1, \ldots, N-2$$
$$\delta(q_{N-1}, 1) = q_{N-1}$$
$$\delta(q_{N-1}, 2) = q_{N-1}$$

It is obvious from the definition of the transition function that q_{N-1} is the dead state and is the only absorbing state. Hence $p(M_N) = 0$. (here we have substituted the name of the machine for the regular set in the argument of p(-), but the meaning is clear.) It is more helpful to look at M_N as described by its transition tree graph given below.



We have labeled the entire tree as T_0 and the N-1 sub-trees as shown. (Henceforth we will omit the machine or regular subscript on λ since it is clear what machine we are talking about.) For $n < N-1$, $\lambda(n) = 2^n$ since all $\alpha \in \{1,2\}^*$ of length less than N-1 are accepted. For $n = N-1$, $\lambda(n) = 2^n-1$ since only $\alpha = 1^{N-1}$ is not accepted. Now for $n > N-1$, since the T_i, $i = 1,2,\ldots,N-1$, are mutually disjoint,

$$\lambda(n) = \lambda_{T_0}(n) = \lambda_{T_1}(n-1) + \lambda_{T_2}(n-2) + \cdots + \lambda_{T_{N-1}}(n-(N-1))$$

where $\lambda_{T_i}(n)$ is the number of strings of length n accepted counting the length from T_i as a base. Furthermore, since each T_i, taken separately, is identical to T_0, each T_i alone recognizes the same set as T_0, which means that $\lambda_{T_i}(n) = \lambda_{T_0}(n) = \lambda(n),\ i=1,\ldots,N-1$ and that

$$\lambda(n) = \lambda(n-1) + \lambda(n-2) + \cdots + \lambda(n-(N-1)), \quad n \geq N-1$$

which we recognize as an N-1 term Fibonacci sequence. For N=3 we get the familiar two term Fibonacci sequence

$$\lambda(0) = 1$$
$$\lambda(1) = 2$$
$$\lambda(2) = 3$$
$$\lambda(3) = 5$$
$$\lambda(4) = 8$$
$$\vdots$$

Because of the type of growth of $\lambda_{M_N}(n)$ we call M_N the <u>N state Fibonacci machine</u>. Since we have a Fibonacci sequence, from Alfred [1], we may calculate

$$\chi = \lim_{n\to\infty} \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)}$$

by solving the polynomial

$$P_N(x) = x^N - x^{N-1} - x^{N-2} - \cdots - x - 1 = 0$$

$P_N(x)$ is continuous and $P_N(0) = -1$, $P_N(2) = 1$, which means that there exists a solution of $P_N(x) = 0$ that is greater than zero and less than two. Furthermore, since for $x > 2$,

$$1 > \frac{1}{x-1} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \sum_{k=1}^{\infty}\left(\frac{1}{x}\right)^k > \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{N-1}}$$

we have

$$x^{N-1} > x^{N-2} + x^{N-3} + \cdots + x + 1$$

$$\Rightarrow N x^{N-1} > N x^{N-2} + N x^{N-3} + \cdots + N x + N$$

$$\Rightarrow N x^{N-1} > (N-1) x^{N-2} + (N-2) x^{N-3} + \cdots + 2x + 1$$

$$\Rightarrow N x^{N-1} - (N-1) x^{N-2} - (N-2) x^{N-3} - \cdots - 2x - 1 > 0$$

$$\Rightarrow P_N'(x) > 0$$

Therefore all solutions of $P_N(x) = 0$ are strictly less than two, which means that

$$\lim_{n\to\infty} \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)} < 2$$

Hence for any k,

$$\lim_{n\to\infty} \frac{\lambda_{M_N}(n+k)}{\lambda_{M_N}(n)} = \lim_{n\to\infty}\left(\frac{\lambda_{M_N}(n+k)}{\lambda_{M_N}(n+k-1)} \cdots \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)}\right)$$

$$= \lim_{n\to\infty} \frac{\lambda_{M_N}(n+k)}{\lambda_{M_N}(n+k-1)} \cdots \lim_{n\to\infty} \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)} < 2^k$$

With the help of this definition and discussion of the N state Fibonacci machine we can prove

Theorem 6.1 For every $\varepsilon > 0$, there exists a finite automaton (and hence a regular set A) such that

$$\frac{\lambda_A(n+1)}{\lambda_A(n)} \longrightarrow \theta$$

where

$$2-\varepsilon < \theta < 2$$

Proof: Since for every $\varepsilon > 0$

$$\sum_{k=1}^{\infty}\left(\frac{1}{2-\varepsilon}\right)^k = \frac{\frac{1}{2-\varepsilon}}{1-\frac{1}{2-\varepsilon}} = \frac{1}{1-\varepsilon} > 1,$$

choose N sufficiently large so that

$$\sum_{k=1}^{N}\left(\frac{1}{2-\varepsilon}\right)^k > 1$$

Form the Fibonacci machine for this N. From the preceding discussion, namely the fact that $\lim_{n\to\infty} \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)} < 2$ we have, for sufficiently large n, $\frac{\lambda_{M_N}(n+1)}{2^{n+1}} < \frac{\lambda_{M_N}(n)}{2^n} < 1$ and hence that $\frac{\lambda_{M_N}(n)}{2^n} \to 0$. Secondly,

$$\sum_{k=1}^{N}\left(\frac{1}{2-\varepsilon}\right)^k > 1 \Rightarrow (2-\varepsilon)^{N-1} + (2-\varepsilon)^{N-2} + \cdots (2-\varepsilon) + 1 > (2-\varepsilon)^N$$

$$\Rightarrow (2-\varepsilon)^N - (2-\varepsilon)^{N-1} - \cdots - (2-\varepsilon) - 1 < 0$$

$$\Rightarrow P_N(2-\varepsilon) < 0$$

Since $P_N(2) = 1 > 0$ for any N (and $P_N(x)$ is continuous)

$$\theta = \lim_{n\to\infty} \frac{\lambda_{M_N}(n+1)}{\lambda_{M_N}(n)} > 2-\varepsilon$$

The Fibonacci machine is especially interesting since it seems that for any other machine, M_A, with N states and such that $\frac{\lambda_A(n)}{2^n} \to 0$, $\lambda_{M_N}(n) \geq \lambda_{M_A}(n)$ for all n. Although not rigorously proved here, this conjecture lends itself to an induction proof on n that would proceed first from the fact that for $n \leq N-1$ the conjecture is obviously true ($\lambda_{M_N}(n) = 2^n$, $n < N-1$ and $\lambda_{M_N}(n) = 2^n - 1$ for $n = N-1$) since the dead state must be accessible in M_A, the other machine, in N-1 or less steps, and then an argument showing successively how M_N must have the minimal number of elements in state q_{N-2}, (q_{N-1} is the dead state), $q_{N-3}, \ldots, q_1$ at each level less than n, and hence the maximal number of elements in state q_0 at each level less than or equal to n, thus implying that at level n + 1, $\lambda_{M_N}(n+1) \geq \lambda_{M_A}(n+1)$. Intuitively speaking this conjecture is not hard to accept since in M_N there is one and only one way to get to the dead state and this path is as long as it can possibly be. Furthermore, all other states besides the dead state are accept states.

With this conjecture in mind, the Fibonacci machine gives a growth rate upper bound for machines which have only one absorbing state, namely the dead state. This growth rate upper bound is seen in theorem 6.1 to be arbitrarily close to two, but always less than two. That is to say, for any machine M which has only the dead state as an absorbing state

$$\lim_{n\to\infty} \frac{\lambda_M(n+1)}{\lambda_M(n)} < 2 .$$

VII. THE SET OF TREES AS AN INPUT

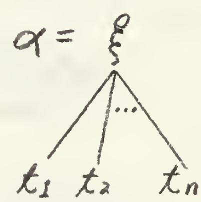
For all automata previously discussed, the input has been a one dimensional tape, i.e., a string of symbols. A tree automaton is essentially the same as an ordinary finite automaton except for the input which is, as the name implies, the set of well formed trees. In order to describe this set we need the following definitions.

Definition 7.1 A ranked alphabet is a pair $\langle A, \sigma \rangle$ where A is a finite set of symbols and $\sigma : A \to \mathbb{N}$, the set of non-negative integers. If we further define

$$A_0 = \{a \in A : \sigma(a) = 0\}$$
$$A_1 = \{a \in A : \sigma(a) = 1\}$$
$$\vdots$$
$$A_k = \{a \in A : \sigma(a) = k\}$$

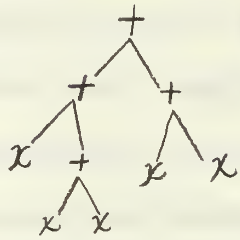
then $A = A_0 \cup A_1 \cup \cdots \cup A_k$

Definition 7.2 α is a tree iff $\alpha \in A_0$ or where $\xi \in A_n$ and $t_i \in A_i$, $i = 0, 1, \ldots, n$



The set of all well-formed trees is a regular set in the sense that it can be recognized by a tree automaton. The question here is: Does there exist a one-one mapping from the set or well-formed trees to the set A* such that the number of times an element of A appears in the tree is the same as the number of times that that particular element appears in the string which is its image and such that the image of the set of trees is a regular set in the sense of definition 2.7? For simplicity, and again without loss of generality, we consider the case where

$A = \{+, x\}$ and $\sigma(+) = 2$, $\sigma(x) = 0$. The condition imposed upon the one-one mapping reduces to saying that if a tree has k +'s and $(k + 1)$ x's (all trees must have one more x than +), then the string which represents that tree must also have k +'s and k + 1 x's. An example of a tree for this A is



Let Y be the set of trees over $\{+, x\}$. Assume ϕ is a mapping which satisfies the above conditions, i.e., $\phi : Y \to \{+, x\}^* \ni T(M) = \phi(Y)$ for some M (we assume M to be minimal) and the strings in $\phi(Y)$ have the correct number of appearances of each element in them. From Knuth [9] we know that the number of trees with n +'s and n + 1 x's is $C_n = \frac{1}{n+1}\binom{2n}{n}$. Let B = $\phi(Y)$. The restrictions on ϕ give us $\lambda_B(2n) = 0$ and $\lambda_B(2n+1) = C_n$ for all n. Using Stirling's approximation to the factorial, i.e., $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we find that $C_n \approx \frac{2^{2n}}{(n+1)\sqrt{\pi n}}$.

Then

$$\frac{\lambda_B(2n+1)}{2^{2n+1}} = \frac{C_n}{2^{2n+1}} \approx \frac{\frac{2^{2n}}{(n+1)\sqrt{\pi n}}}{2^{2n+1}} \to 0$$

and

$$\frac{\lambda_B(2n)}{2^{2n}} = \frac{0}{2^{2n}} \to 0$$

Therefore,

$$\frac{\lambda_B(n)}{2^n} \to 0 \Rightarrow \rho(B) = 0$$

M_B must then be a machine with only one absorbing state, namely the dead state. But if we look at the sequence

$$\frac{\lambda_B(2n+1)}{\lambda_B(2n-1)} = \frac{C_n}{C_{n-1}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\frac{1}{n}\binom{2n-2}{n-1}}$$

$$= \frac{4n^2 - 2n}{n^2 + n} \longrightarrow 4$$

we see that the growth rate is too large for a machine of this type.

Hence, a contradiction and so no such mapping exists.

VIII. SUMMARY AND CONCLUSIONS

Looking back now on the problem of section 7, we can see why something more than the criteria given in section 3 was needed to solve it. First of all, so little was known about how the supposedly regular set B behaved that it would have been difficult to decide even if $\lim_{n \to \infty} \frac{\pi_B(n)}{n}$ existed. Secondly, although theorem 3.7 could be applied, it was not a sharp enough criterion to give any useful information. Essentially all that was known was that $\lambda_B(2n) = 0$ and $\lambda_B(2n+1) = C_n$. However, with this new concept of p(B), and knowing theorems 4.4 and 5.2, we could deduce information about what type of machine would be needed to recognize B, namely a machine with a single absorbing state -- the dead state. Next, with the Fibonacci machine providing a growth rate upper bound for this particular type of machine we could show that $\lambda_B(n)$ grew too fast for this type machine and hence B could not be regular. This problem was especially difficult, since the growth rate of $\lambda_B(n)$ was just barely greater than allowable and a really fine line had to be drawn to mark the cutoff point. Therefore, p(-) is an improvement over $\lim_{n \to \infty} \frac{\pi(n)}{n}$ in that it is a more specific indicator giving more information about the machine, namely not only the existence of a dead state, but that the dead state is the only absorbing state. Coupling this with the "maximality" of the Fibonacci machine yields another criterion for showing a set to be non-regular.

BIBLIOGRAPHY

1. Alfred, Brother U., An Introduction to Fibonacci Discovery, p. 12, The Fibonacci Association, 1965.

2. Cainaiello, E. R., ed., Automata Theory, p. 304-313, Academic Press, 1966.

3. Feller, William, An Introduction to Probability Theory and Its Applications, 2d ed., v. 1, p. 355, Wiley, 1957.

4. Finkbeiner, Daniel T., Introduction to Matrices and Linear Transformations, 2d ed., p. 239, W. H. Freeman and Company, 1966.

5. Fisz, Marek, Probability Theory and Mathematical Statistics, 3d ed., p. 256, Wiley, 1963.

6. Harrison, M. A., Introduction to Switching and Automata Theory, McGraw-Hill, 1965.

7. Kemeny, John G., and others, Finite Mathematical Structures, p. 404-408, Prentice Hall, 1959.

8. Kemeny, J. G. and Snell, J. L., Finite Markov Chains, p. 22, D. Van Nostrand Company, 1960.

9. Knuth, Donald E., The Art of Computer Programming: Fundamental Algorithms, v. 1, p. 389, Addison Wesley, 1968.

10. Minsky, M. and Papert, S., "Unrecognizable Sets of Numbers," Journal of the Association for Computing Machinery, v. 13, p. 281-286, April 1966.

11. Moore, Edward F., ed., Sequential Machines: Selected Papers, p. 63-91, Addison-Wesley, 1964.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Naval Ship Systems Command (Code 2052) Department of the Navy Washington, D. C. 20370	1
4.	Asst. Professor W. S. Brainerd, Code 53Bz Department of Mathematics Naval Postgraduate School Monterey, California 93940	2
5.	LTJG Ronald B. Knode, USN 909 South Washington St., Apt 113 Alexandria, Virginia 22314	1

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE On the Unrecognizability of Sets	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Master's Thesis; (June 1969)	
5. AUTHOR(S) (First name, middle initial, last name) Ronald Barry Knode	

6. REPORT DATE June 1969	7a. TOTAL NO. OF PAGES 42	7b. NO. OF REFS 11
8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT ~~Distribution of this document is unlimited.~~ This document has been approved for public release and sale; its distribution is unlim.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940	

13. ABSTRACT

Let $M = \langle Q, q_0, \delta, F \rangle$ be a finite automaton over the alphabet $\Sigma = \{1,\dots,k\}$. A state $q \in Q$ is a dead state iff $q \notin F$ and $\delta(q,\alpha) = q \ \forall \alpha \in \Sigma^*$. Let - be a mapping from Σ^* onto the non-negative integers defined by $\bar{\Lambda} = 0$ (Λ is the empty string) $\overline{\alpha\tau} = k\bar{\alpha} + \bar{\tau}, \tau \in \Sigma, \alpha \in \Sigma^*$. Define $\pi_A(n) = \#\{\alpha \in A : 0 \le \bar{\alpha} \le n\}$ and $\lambda_A(n) = \#\{\alpha \in A : \ell(\alpha) = n\}$. If A is regular let M_A be the minimal automaton recognizing A. Each automaton M induces a Markov process obtained by considering the inputs to be generated by independent rolls of a k-sided fair die. Let p(M) represent the probability of being in a final state. Let $p(A) = p(M_A)$. The following are proved: 1) $\frac{\pi_A(n)}{n} \to \theta \Rightarrow \frac{\lambda_A(n)}{2^n} \to \theta$; 2) $\frac{\pi_A(n)}{n} \to \theta$, A regular $\Rightarrow p(A) = \theta$; 3) $p(A) = 0 \Rightarrow M_A$ has the dead state as the only absorbing state; 4) $\forall \varepsilon > 0 \ \exists$ a regular set $A \ni M_A$ has a dead state and $\frac{\lambda_A(n+1)}{\lambda_A(n)} \to \theta$ where $k - \varepsilon < \theta < k$; 5) If $p(A) = 0$, then $\frac{\lambda_A(n+1)}{\lambda_A(n)}$ cannot converge to k. With $k = 2$, these results prove that there is no regular set A such that $\lambda_A(2n+1) = \frac{1}{n+1}\binom{2n}{n}$ and $\lambda_A(2n) = 0$. Hence there is no 1-1 mapping from the set of all trees representing expressions involving a binary + and a variable x into $\{+, x\}^*$ which preserves the number of +'s and x's and such that the set of tree images is a regular set.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Unrecognizability						
Fibonacci machine						
Markov process						
Automata						
Regular set						